The averaging of weakly nonlocal Symplectic Structures.

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Abstract

We consider the averaging of the weakly nonlocal Symplectic Structures corresponding to local evolution PDE's in the Whitham method. The averaging procedure gives the weakly nonlocal Symplectic Structure of Hydrodynamic Type for the corresponding Whitham system. The procedure gives also the "action variables" corresponding to the wave numbers of m-phase solutions of initial system which give the additional conservation laws for the Whitham system.

We consider the general weakly nonlocal Symplectic Structures ([1]) having the form

$$\Omega_{ij}(x,y) = \sum_{k\geq 0} \omega_{ij}^{(k)}(\varphi,\varphi_x,\ldots)\delta^{(k)}(x-y) + \sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta\varphi^i(x)}\,\nu(x-y)\,\frac{\delta H^{(s)}}{\delta\varphi^j(y)} \qquad (1)$$

where $i,j = 1,\ldots,n$, $\nu(x-y) = (1/2)sgn(x-y)$, $\delta^{(k)}(x-y) = \partial^k/\partial x^k \delta(x-y)$, $e_s = \pm 1$, $H^{(s)}[\varphi] = \int_{-\infty}^{+\infty} h^{(s)}(\varphi,\varphi_x,\ldots)dx$ and $\Omega_{ij}(x,y)$ gives the closed 2-form on the space of functions $\varphi(x) = (\varphi^1(x),\ldots,\varphi^n(x))$. We assume also that every term in (1) depends on the finite number of derivatives $\varphi,\varphi_x,\ldots$. [1]

We say that the evolution system

$$\varphi^i_t \;\; = \;\; Q^i(\varphi,\varphi_x,\ldots) \qquad (2)$$

admits the Symplectic Structure (1) with the Hamiltonian function $H = \int_{-\infty}^{+\infty} h(\varphi,\varphi_x,\ldots)dx$ if the following relation is true

$$\sum_{k\geq 0} \omega_{ij}^{(k)}(\varphi,\varphi_x,\ldots)\frac{\partial^k}{\partial x^k}Q^j(\varphi,\varphi_x,\ldots) + \sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta\varphi^i(x)} D^{-1} \frac{\delta H^{(s)}}{\delta\varphi^j(x)} Q^j(\varphi,\varphi_x,\ldots) \equiv \frac{\delta H}{\delta\varphi^i(x)}$$

[1] Easy to see that the nonlocal part of (1) is connected actually with the quadratic form on the space of functionals $H^{(s)}$ which is written here in the diagonal form.

where D^{-1} is the integration operator defined in the skew-symmetric way, i.e. $D^{-1}\xi(x) = (1/2)\int_{-\infty}^{x}\xi(y)dy - (1/2)\int_{x}^{+\infty}\xi(y)dy$. The functionals $H^{(s)}[\varphi]$ should give the conservation laws for the system (2) in this case such that $h_t^{(s)} \equiv \partial_x J^{(s)}$ for some $J^{(s)}(\varphi, \varphi_x, \dots)$.

We are going to consider the well-known Whitham averaging method [2,3,4,5,6] for system (2) which gives the nonlinear system of equations on the slow-modulated parameters U^{ν} of m-phase solutions of (2)

$$\varphi^i(x,t) \quad = \quad \Phi^i(\omega(U)t + k(U)x + \theta_0, U^1, \dots, U^N) \tag{3}$$

having the form

$$U^{\nu}_T \quad = \quad V^{\nu}_{\mu}(U)\; U^{\mu}_X \;\;, \quad \nu, \mu = 1, \dots, N \tag{4}$$

(Whitham system). The functions $\Phi^i(\theta, U)$ here are 2π-periodic functions with respect to the variables $(\theta^1, \dots, \theta^m) = \theta$ depending on the additional parameters $U^1, \dots, U^N$. The (m-component) vectors $\omega(U)$ and $k(U)$ play the role of frequences and "wave numbers" for the quasiperiodic solutions (3). In Whitham method the parameters U^{ν} become the functions of the "slow variables" $X = \epsilon x$, $T = \epsilon t$, $\epsilon \to 0$ and the system (4) gives the necessary conditions for the construction of corresponding asymptotic solution of (2).

We will give here the procedure of "averaging"of the Symplectic Structure (1) giving the weakly nonlocal Symplectic Structure of Hydrodynamic Type for system (4).

Definition 1. *We call the weakly nonlocal Symplectic Structure of Hydrodynamic Type the weakly nonlocal Symplectic Structure having the form*

$$\Omega_{\nu\mu}(X,Y) = \sum_{s,p=1}^{M} \kappa_{sp} \frac{\partial f^{(s)}}{\partial U^{\nu}}(X)\; \nu(X-Y)\; \frac{\partial f^{(p)}}{\partial U^{\mu}}(Y) \tag{5}$$

where $f^{(s)}(U)$ are some functions of variables $U^1, \dots, U^N$ and κ_{sk} is some constant symmetric bilinear form ($\nu, \mu = 1, \dots, N$).

Let us formulate now the procedure giving the Symplectic Structure (5) for the Whitham system. We introduce first the "averaging" of any local function $f(\varphi, \varphi_x, \dots)$ over the "invariant tori" corresponding to the family (3). Namely

$$\langle f \rangle(U) = \int_0^{2\pi} \dots \int_0^{2\pi} f\,(\,\Phi(\theta,U),\; k^{\alpha}(U)\,\Phi_{\theta^{\alpha}}(\theta,U), \dots)\;\; \frac{d^m\theta}{(2\pi)^m}$$

Let us introduce the "extended functional space" of functions $\varphi(\theta, x)$ 2π-periodic w.r.t. to each θ^{α} at every fixed x and define the "extended" Symplectic Form $\tilde{\Omega}_{ij}(\theta, \theta', x, y) = \delta(\theta - \theta')\Omega_{ij}(x,y)$.

We introduce also the special functions $T_{\alpha}^{(s)}$ connected with the nonlocal part of the Symplectic Structure $\tilde{\Omega}_{ij}(\theta,\theta',x,y)$

$$T_{\alpha}^{(s)}(\varphi,\varphi_x,\varphi_{\theta^\alpha},\ldots) = \sum_{k\geq 1}\sum_{p=0}^{k-1}(-1)^p \left(\frac{\partial h^{(s)}}{\partial \varphi^i_{kx}}\right)_{px} \varphi_{\theta^\alpha,(k-p-1)x} \tag{6}$$

(here $f_{kx} \equiv \partial^k f/\partial x^k$ and we assume summation over the repeated indices).

Lemma 1.

1) For any Symplectic Form (1) we have the relations

$$\varphi^i_{\theta^\alpha} \sum_{k\geq 0} \omega_{ij}^{(k)}(\varphi,\varphi_x,\ldots)\varphi^j_{\theta^\beta,kx} + \sum_{s=1}^{g} e_s \left(h^{(s)}_{\theta^\beta} T^{(s)}_{\alpha} - h^{(s)}_{\theta^\alpha} T^{(s)}_{\beta} + \left(T^{(s)}_{\alpha}\right)_x T^{(s)}_{\beta}\right) \equiv$$

$$\equiv \frac{\partial}{\partial \theta^\gamma} Q^{\gamma}_{\alpha\beta}(\varphi,\ldots) + \frac{\partial}{\partial x} A_{\alpha\beta}(\varphi,\ldots)$$

for some functions $Q^{\gamma}_{\alpha\beta}(\varphi,\ldots)$, $A_{\alpha\beta}(\varphi,\ldots)$

2) The functions $A_{\alpha\beta}(\varphi,\ldots)$ (defined modulo constant functions) can be normalized in such a way that $A_{\alpha\beta}(\varphi,\ldots) \equiv 0$ for any $\varphi(\theta,x)$ depending on x only (and constant with respect to θ at every fixed x).

We will assume now that we have m linearly independent commuting flows

$$\varphi^i_{t^\alpha} \;\; = \;\; Q^i_{\alpha}(\varphi,\varphi_x,\ldots) \;, \quad \alpha = 1,\ldots,m$$

for the system (2) which leave the family (3) invariant generating the linear shifts of initial phases θ_0^α on it with the frequences $\omega_{(\alpha)}(U)$. We require also that the functionals $H^{(s)}[\varphi]$ give the conservation laws for the flows $\varphi^i_{t^\alpha}$ such that

$$h^{(s)}_{t^\alpha} \;\; \equiv \;\; \partial_x J^{(s)}_{\alpha} \tag{7}$$

for some local functions $J^{(s)}_{\alpha}(\varphi,\varphi_x,\ldots)$.

Besides that, we will require that the matrix $||\omega^{\beta}_{(\alpha)}(U)||$ is nondegenerate and has the inverse matrix $||\gamma^{\beta}_{\alpha}(U)||$ such that

$$\gamma^{\delta}_{\alpha}(U)\; \omega^{\beta}_{(\delta)}(U) \;\; \equiv \;\; \delta^{\beta}_{\alpha} \tag{8}$$

Theorem 1.

Under the conditions formulated above the Whitham system (4) has the weakly nonlocal Symplectic Structure of Hydrodynamic Type

$$\Omega_{\nu\mu}(X,Y) = \sum_{\alpha=1}^{m} \left(\frac{\partial k^\alpha}{\partial U^\nu}(X)\; \nu(X-Y)\; \frac{\partial I_\alpha}{\partial U^\mu}(Y) + \frac{\partial I_\alpha}{\partial U^\nu}(X)\; \nu(X-Y)\; \frac{\partial k^\alpha}{\partial U^\mu}(Y)\right) +$$

$$+\sum_{s=1}^{g} e_s \frac{\partial\langle h^{(s)}\rangle}{\partial U^{\nu}}(X)\;\nu(X-Y)\;\frac{\partial\langle h^{(s)}\rangle}{\partial U^{\mu}}(Y) \tag{9}$$

with Hamiltonian function $H=\int_{-\infty}^{+\infty}\langle h\rangle dX$ where the "action variables" $I_\alpha(U)$ are given through the formulas:

$$\begin{aligned}
\frac{\partial I_\alpha}{\partial U^\nu} \;=\; \frac{\partial k^\beta}{\partial U^\nu}\Bigg[&-\langle A_{\alpha\beta}\rangle + \gamma_\alpha^\delta \sum_{s=1}^{g} e_s \left(\langle T_\beta^{(s)} J_\delta^{(s)}\rangle - \langle T_\beta^{(s)}\rangle\langle J_\delta^{(s)}\rangle\right) \;- \\
&- \frac{1}{2}\gamma_\alpha^\delta\gamma_\beta^\zeta \sum_{s=1}^{g} e_s \left(\langle J_\delta^{(s)} J_\zeta^{(s)}\rangle - \langle J_\delta^{(s)}\rangle\langle J_\zeta^{(s)}\rangle\right)\Bigg] + \\
&+\langle \Phi^i_{U^\nu} \sum_{k\geq 0} \omega_{ij}^{(k)}(\varphi,\dots)\varphi^j_{\theta^\alpha,kx}\rangle - \sum_{s=1}^{g} e_s \langle \Phi^i_{U^\nu}\frac{\delta H^{(s)}}{\delta\varphi^i(x)} T_\alpha^{(s)}\rangle + \\
&+\sum_{s=1}^{g} e_s\gamma_\alpha^\beta \left(\langle \Phi^i_{U^\nu}\frac{\delta H^{(s)}}{\delta\varphi^i(x)} J_\beta^{(s)}\rangle - \langle \Phi^i_{U^\nu}\frac{\delta H^{(s)}}{\delta\varphi^i(x)}\rangle\langle J_\beta^{(s)}\rangle\right)
\end{aligned} \tag{10}$$

and the functions $A_{\alpha\beta}$ are normalized according to Lemma 1.

Definition 2. *We call the form (9) the averaging of the weakly nonlocal Symplectic Structure (1) on the family of m-phase solutions of system (2).*

Let us add also that the functionals $\int_{-\infty}^{+\infty} I_\alpha dX$ as well as $\int_{-\infty}^{+\infty} k^\alpha dX$, $\int_{-\infty}^{+\infty}\langle h^{(s)}\rangle dX$ and $H=\int_{-\infty}^{+\infty}\langle h\rangle dX$ give the conservation laws for the Whitham system (4).

We will give now another variant of the averaging procedure of the form (1) using the averaging of weakly nonlocal 1-forms on the space $\varphi(x)$.

Let us consider the 1-forms $\omega_i[\varphi](x)$ on the space of functions $\varphi^i(x)$, $i=1,\dots,n$ having the form

$$\omega_i[\varphi](x) \;=\; c_i(\varphi,\varphi_x,\dots) \;-\; \frac{1}{2}\sum_{s=1}^{g} e_s \frac{\delta H^{(s)}}{\delta\varphi^i(x)} \int_{-\infty}^{+\infty} \nu(x-y)\, h^{(s)}(\varphi,\varphi_y,\dots)\, dy \tag{11}$$

where $H^{(s)}[\varphi] \;=\; \int_{-\infty}^{+\infty} h^{(s)}(\varphi,\varphi_x,\dots)\, dx$.

The action of the forms $\omega_i[\varphi](x)$ on the "tangent vectors" $\xi^i[\varphi](x)$ is defined in the natural way

$$(\boldsymbol{\omega}, \boldsymbol{\xi})[\varphi] \; = \; \int_{-\infty}^{+\infty} \omega_i[\varphi](x) \, \xi^i[\varphi](x) \, dx$$

The forms (11) are closely connected with the weakly nonlocal 2-forms (1). Namely, let us consider the external derivative of the form $\omega_i[\varphi](x)$:

$$[d\boldsymbol{\omega}]_{ij}(x, y) \; = \; \frac{\delta \omega_j[\varphi](y)}{\delta \varphi^i(x)} \; - \; \frac{\delta \omega_i[\varphi](x)}{\delta \varphi^j(y)}$$

Lemma 2.
The external derivative $[d\boldsymbol{\omega}]_{ij}(x, y)$ is the closed 2-form having the form (1) with some local functions $\omega_{ij}^{(k)}(\varphi, \varphi_x, \dots)$.

Let us formulate also the opposite statement

Lemma 3.
Every closed 2-form (1) can be locally represented as the external derivative of some 1-form (11) on the space $\varphi(x)$.

We are going to give now the procedure of averaging of 1-forms (11) connected with the averaging of the Symplectic structures (1). Namely, we will assume now that the form $\Omega_{ij}(x, y)$ is represented as the external derivative of the form (11). The corresponding procedure of averaging of the form (11) should then give the weakly nonlocal 1-form of "Hydrodynamic type" which is connected with the form $\Omega_{\nu\mu}(X, Y)$ in the same way.

Definition 3. *We call the form $\omega_\nu[\mathbf{U}](X)$ on the space of functions $U^1(X), \dots, U^N(X)$ the weakly nonlocal 1-form of Hydrodynamic type if it has the form*

$$\omega_\nu[\mathbf{U}](X) \; = \; -\frac{1}{2} \sum_{s,p=1}^{M} \kappa_{sp} \, \frac{\partial f^{(s)}}{\partial U^\nu}(\mathbf{U}(X)) \int_{-\infty}^{+\infty} \nu(X - Y) \, f^{(p)}(\mathbf{U}(Y)) \, dY$$

for some functions $f^{(s)}(\mathbf{U})$ and the quadratic form κ_{sp}.

Definition 4. *We call the 1-form*

$$\omega_\nu(X) \; = \; - \frac{\partial k^\alpha}{\partial U^\nu}(X) \int_{-\infty}^{+\infty} \nu(X - Y) \, I_\alpha(Y) \, dY \; -$$

$$- \; \frac{1}{2} \sum_{s=1}^{g} e_s \, \frac{\partial \langle h^{(s)} \rangle}{\partial U^\nu}(X) \int_{-\infty}^{+\infty} \nu(X - Y) \, \langle h^{(s)} \rangle(Y) \, dY \qquad (12)$$

where $I_\alpha(\mathbf{U})$ are defined by the formulas

$$I_\alpha(\mathbf{U}) \;=\; \langle c_i\, \varphi^i_{\theta^\alpha} \rangle \,+\, \frac{1}{2}\, \gamma^\delta_\alpha(\mathbf{U}) \sum_{s=1}^{g} e_s \left[\langle h^{(s)}\, J^{(s)}_\delta \rangle \,-\, \langle h^{(s)} \rangle \langle J^{(s)}_\delta \rangle \right] \,-\, \frac{1}{2} \sum_{s=1}^{g} e_s\, \langle h^{(s)}\, T^{(s)}_\alpha \rangle \tag{13}$$

- the averaging of the 1-form (11) on the family of m-phase solutions of (2).

Theorem 2.
The averaged forms $\Omega_{\nu\mu}(X,Y)$ and $\omega_\nu(X)$ are connected by the relation

$$\Omega_{\nu\mu}(X,Y) \;=\; [d\,\boldsymbol{\omega}]_{\nu\mu}(X,Y) \;=\; \frac{\delta \omega_\mu(Y)}{\delta U^\nu(X)} \;-\; \frac{\delta \omega_\nu(X)}{\delta U^\mu(Y)}$$

The quantities $I_\alpha(\mathbf{U})$ defined in (13) coincide with the same quantities defined in (10).

It's not difficult to see also that the form (12) differs from the form

$$\omega'_\nu(X) = -\frac{1}{2}\, \frac{\partial k^\alpha}{\partial U^\nu}(X) \int_{-\infty}^{+\infty} \nu(X-Y)\, I_\alpha(Y)\, dY \;-\; \frac{1}{2}\, \frac{\partial I_\alpha}{\partial U^\nu}(X) \int_{-\infty}^{+\infty} \nu(X-Y)\, k^\alpha(Y)\, dY \,-$$

$$-\;\frac{1}{2} \sum_{s=1}^{g} e_s\, \frac{\partial \langle h^{(s)} \rangle}{\partial U^\nu}(X) \int_{-\infty}^{+\infty} \nu(X-Y)\, \langle h^{(s)} \rangle (Y)\, dY$$

just by exact 1-form

$$\boldsymbol{\omega} \;-\; \boldsymbol{\omega}' \;=\; d\; \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I_\alpha(X)\, \nu(X-Y)\, k^\alpha(Y)\; dX\, dY$$

The formulas (12), (13) give another procedure for the averaging of 2-forms $\Omega_{ij}(x,y)$ represented in the form of the external derivatives of weakly-nonlocal 1-forms $\omega_i(x)$.

We can also write the formal (nonlocal) Lagrangian formalism for the Whitham equations in the form

$$\delta \int \left[\omega_\nu(X)\, U^\nu_T(X) \;-\; \langle h \rangle (\mathbf{U}) \right]\; dX\, dT \;=\; 0$$

The work was supported by Grant of President of Russian Federation (MK - 1375.2003.02), Russian Foundation for Basic Research (grant RFBR 03-01-00368), and Russian Science Foundation.

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